TRINITY COLLEGE

Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS METHODS UNIT 3

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(5 marks)**

Fuel flows into a storage tank that is initially empty at a rate of $\sqrt{4+3t}$ litres per minute, where t is the time in minutes and $0 \le t \le 100$.

(a) Determine how much fuel is in the tank after 20 minutes. (2 marks)

Solution
$V = \int_0^{20} \sqrt{4+3t}\,dt = 112\text{ L}$
Specific behaviours
✓ writes integral ✓ evaluates integral

(b) If the tank is completely full after 100 minutes, determine the time required for the tank to become one-quarter full. (3 marks)

Solution
$V = \int_0^{100} \sqrt{4+3t}\,dt = 1176.09\text{ L}$ $\int_0^{T} \sqrt{4+3t}\,dt = \frac{1176.09}{4}$ $\frac{2}{9}\left(\sqrt{4+3T}\right)^{\frac{3}{2}} - \frac{16}{9} = 294.02 \Rightarrow T = 39.0\text{ minutes}$
Specific behaviours
✓ calculates total volume ✓ writes integral and equates to quarter volume ✓ evaluates time

Question 10 **(8 marks)**

The potential difference, V volts, across the terminals of an electrical capacitor t seconds after it begins to discharge through a resistor can be modelled by the equation

$$V = V_0 e^{-kt}$$

V_0 is the initial potential difference and k is a constant that depends on the size of the capacitor and the resistor.

(a) If $V_0 = 15.8$ volts and $k = 0.013$, determine

(i) the potential difference across the capacitor 2 minutes after discharge began. (2 marks)

Solution
When $t = 120, V = 3.32$ volts
Specific behaviours
✓ uses correct time ✓ calculates correct voltage

(ii) the time taken for the potential difference to drop from 10.5 to 7.5 volts. (3 marks)

Solution
When $V = 10.5, t = 31.4$ and when $V = 7.5, t = 57.3$. Hence takes $57.3 - 31.4 = 25.9$ seconds.
Specific behaviours
✓ calculates first time ✓ calculates second time ✓ calculates difference, correct to at least 1 dp

(iii) the rate of change of V when the potential difference is 5 volts. (1 mark)

Solution
$\frac{dV}{dt} = -kV = -0.013 \times 5 = -0.065$ volts/sec
Specific behaviours
✓ calculates rate

(b) Another capacitor takes 110 seconds for its maximum potential difference to halve. It is instantly recharged to its maximum every 4 minutes, which is the time required for the potential difference to fall from its maximum to 3.5 volts. Determine the maximum potential difference for this capacitor. (2 marks)

Solution
$e^{-110k} = 0.5 \Rightarrow k = 0.0063$ $3.5 = V_0 e^{-0.0063\times 240} \Rightarrow V_0 = 15.88$
Specific behaviours
✓ determines k ✓ determines V_0

Question 11 **(7 marks)**

X is a uniform discrete random variable where $x = 1, 2, 3, 4, 6, 8, 11$.

(a) Determine

(i) $P(X \geq 3)$. (1 mark)

Solution
$P(X \geq 3) = \frac{5}{7}$
Specific behaviours
✓ correct value

(ii) $P(X > 2 \mid X \leq 8)$. (2 marks)

Solution
$P(X \leq 8) = \frac{6}{7}$ $P(X > 2 \vert X \leq 8) = \frac{4}{7} \div \frac{6}{7} = \frac{2}{3}$
Specific behaviours
✓ $P(X \leq 8)$ ✓ correct probability

(b) Calculate the value of

(i) $\mathrm{E}(X)$. (2 marks)

Solution
$\mathrm{E}(X) = \frac{1+2+3+4+6+8+11}{7}$ $= 5$
Specific behaviours
✓ expression   ✓ $\mathrm{E}(X)$

(ii) $\mathrm{SD}(X)$. (2 marks)

Solution
$\sigma_X = \frac{2\sqrt{133}}{7} \approx 3.295$
Specific behaviours
✓ standard deviation   ✓ $\mathrm{Var}(X)$

Question 12 **(8 marks)**

A manufacturing process begins and the rate at which it produces gas after t minutes ($t \geq 0$) is modelled by

$$r(t) = 62.5(1 - e^{-0.16t})\ \text{m}^3/\text{minute}$$

(a) State the maximum rate that gas can be produced at. (1 mark)

Solution
$62.5\ \text{m}^3/\text{minute}$
Specific behaviours
✓ correct rate

(b) Calculate the rate that gas is being produced after 5 minutes. (1 mark)

Solution
$r(1) = 62.5(1 - e^{-0.8}) = 34.42\ \text{m}^3/\text{minute}$
Specific behaviours
✓ correct rate (exact or at least 1dp)

(c) Use the increments formula to determine the approximate change in r between 120 and 125 seconds after production began. (3 marks)

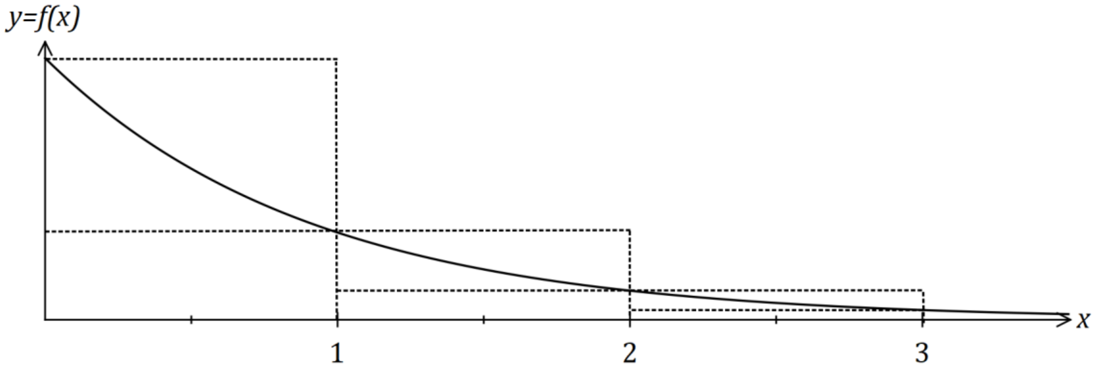
Solution
$\delta r \approx \frac{dr}{dt}\delta t \approx 10e^{-0.16t} \times \delta t$ $\approx \frac{10}{e^{0.32}} \times \frac{5}{60}$ $\approx \frac{5}{6e^{0.32}} \approx 0.605\ \text{m}^3/\text{minute}$
Specific behaviours
✓ correct $r'(2)$ ✓ correct δt ✓ correct change

(d) Use the increments formula to determine the approximate volume of gas produced in the 12 seconds following $t = 5$. (3 marks)

Solution
$\delta V \approx \frac{dV}{dt}\delta t$ $\approx r(t) \times \delta t$ $\approx 34.42 \times \frac{12}{60}$ $\approx 6.883\ \text{m}^3$
Specific behaviours
✓ correct use of increments formula   ✓ uses correct t and δt ✓ correct estimate (at least 2dp)

Question 13 **(5 marks)**

The function $f(x) = \dfrac{3}{3^x}$ is shown below.



(a) Use the sum of the areas of the circumscribed rectangles shown in the diagram to explain why $\int_0^3 f(x)\,dx < \dfrac{13}{3}$. (2 marks)

Solution
$$\text{Sum} = 1 \times \left(3 + 1 + \frac{1}{3}\right) = \frac{13}{3}$$ Since the definite integral represents area under curve, the value of the integral must be less than $\frac{13}{3}$.
Specific behaviours
✓ shows calculation for area overestimate ✓ explains area under curve must be less than overestimate

(b) Use the average of the sum of the areas of the inscribed rectangles and the sum of the areas of the circumscribed rectangles shown to determine an estimate for $\int_0^3 f(x)\,dx$. (2 marks)

Solution
$$\text{Sum}_2 = 1 \times \left(1 + \frac{1}{3} + \frac{1}{9}\right) = \frac{13}{9}$$ $$\text{Avg} = \frac{13}{3} + \frac{13}{9} = \frac{26}{9} \text{ sq units}$$
Specific behaviours
✓ shows calculation for underestimate ✓ calculates average

(c) Suggest a modification to the method used in (b) to achieve a better estimate for $\int_0^3 f(x)\,dx$. (1 mark)

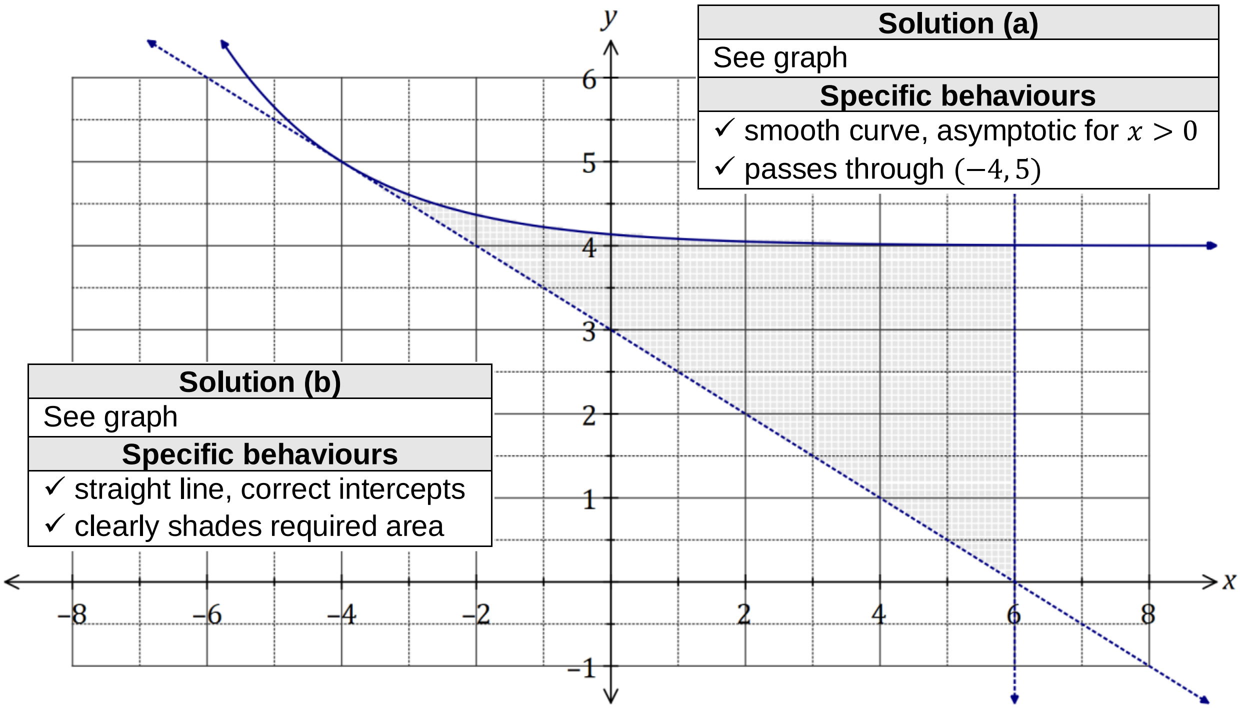
Solution
Use a larger number of narrower rectangles.
Specific behaviours
✓ sensible modification

Question 14 **(7 marks)**

Let $f(x) = 4 + e^{-0.5x-2}$.

(a) Sketch the graph of $y = f(x)$ on the axes below. (2 marks)

Solution (a)
See graph
Specific behaviours
✓ smooth curve, asymptotic for $x > 0$
✓ passes through $(-4, 5)$



Solution (b)
See graph
Specific behaviours
✓ straight line, correct intercepts
✓ clearly shades required area

(b) The line $y = 3 - 0.5x$ is a tangent to the curve $y = f(x)$ at $x = -4$, and it intersects the x-axis at the point $(k, 0)$. Add the line to the graph above and shade the area enclosed by the line, the curve and $x = k$. (2 marks)

(c) Determine the area enclosed by the line, the curve and $x = k$. (3 marks)

Solution
$3 - 0.5k = 0 \Rightarrow k = 6$ $A = \int_{-4}^{6} (4 + e^{-0.5x-2}) - (3 - 0.5x)dx$ $= 17 - 2e^{-5} \approx 16.99 \text{ sq units}$
Specific behaviours
✓ indicates value of k ✓ writes integral using difference of functions   ✓ evaluates integral

Question 15 **(12 marks)**

The random variable X is the number of goals scored by a team in a soccer match, where

$$P(X = x) = \frac{1.9^x e^{-1.9}}{x!} \quad \text{for } x = 0, 1, 2, 3, \dots \text{to infinity}$$

(a) Determine the probability that the team scores at least one goal in a match. (2 marks)

Solution
$P(X = 0) = 0.1496$ $P(X > 0) = 1 - 0.1496 = 0.8504$
Specific behaviours
✓ $P(X = 0)$ ✓ correct probability

The random variable Y is the bonus each player is paid after a match, depending on the number of goals the team scored. For one or two goals $\$150$ is paid, for three goals $\$300$ is paid and for four or more goals $\$600$ is paid. No bonus is paid if no goals are scored.

(b) Complete the probability distribution table for Y. (3 marks)

Goals scored	$x = 0$	$1 \le x \le 2$	$x = 3$	$x \ge 4$
y ($\$$)	0	$\mathbf{150}$	$\mathbf{300}$	600
$P(Y = y)$	$\mathbf{0.1496}$	$\mathbf{0.5542}$	$\mathbf{0.1710}$	0.1252

Solution
$P(Y = 300) = P(X = 3) = 0.1710$ $P(Y = 150) = 1 - 0.1496 - 0.1710 - 0.1252 = 0.5542$
Specific behaviours
✓ missing y values ✓ $P(Y = 0)$ and $P(Y = 300)$ ✓ $P(Y = 150)$

(c) Calculate

(i) the mean bonus paid per match. (2 marks)

Solution
$\bar{Y} = 0 + 83.13 + 51.30 + 75.12$ $= \$209.55$
Specific behaviours
✓ expression ✓ mean

(ii) the standard deviation of the bonus paid per match. (2 marks)

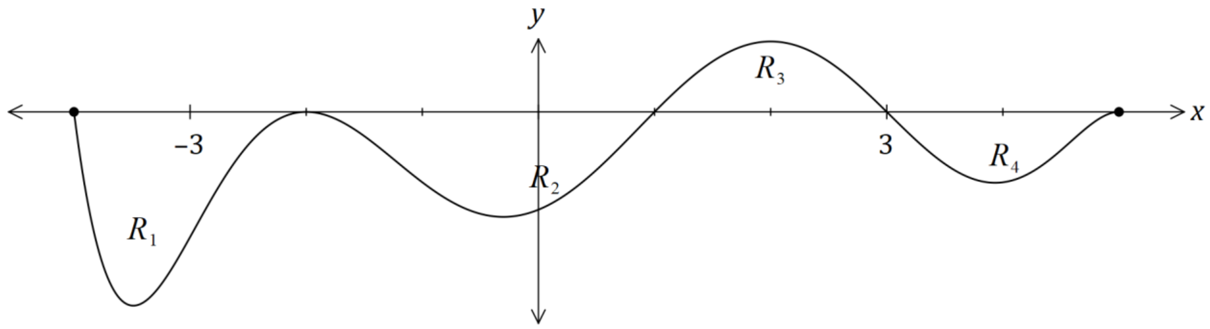
Solution
$\sigma_Y^2 = 29020.3$ $\sigma_Y = \$170.35$
Specific behaviours
✓ variance ✓ standard deviation

(d) The owner of the team plans to increase the current bonuses by $\$90$ next season (so that the players will get a bonus of $\$90$ even when no goals are scored) and then further raise them by 18% the following season. Determine the mean and standard deviation of the bonus paid per match after both changes are implemented. (3 marks)

Solution
$Z = (Y + 90) \times 1.18$ $\bar{Z} = (209.55 + 90) \times 1.18 = \$353.47$ $\sigma_Z = 170.35 \times 1.18 = \$201.02$
Specific behaviours
✓ correct multiplier ✓ new mean ✓ new standard deviation

Question 16 **(7 marks)**

The graph of $y = f(x)$ is shown below for $-4 \le x \le 5$.



The area trapped between the x-axis and the curve for regions R_1, R_2, R_3 and R_4 are $35, 52, 28$ and 24 square units respectively.

(a) Determine the value of

(i) $\displaystyle\int_{-4}^{-2} f(x)\,dx$. (1 mark)

Solution
-35
Specific behaviours
✓ correct value

(ii) $\displaystyle\int_{-2}^{5} f(x)\,dx$. (2 marks)

Solution
$-52 + 28 - 24 = -48$
Specific behaviours
✓ shows sum of signed areas
✓ correct value

(iii) $\displaystyle\int_{1}^{5} (f(x) - 7)\,dx$. (2 marks)

Solution
$(28 - 24) - 4 \times 7 = 4 - 28 = -24$
Specific behaviours
✓ area of rectangle
✓ correct value

(iv) $\displaystyle\int_{-4}^{1} f(x)\,dx - \int_{1}^{5} f'(x)\,dx$. (2 marks)

Solution
$-35 - 52 - (0 - 0) = -87$
Specific behaviours
✓ shows first integral is zero
✓ correct value

Question 17 **(9 marks)**

Seeds were planted in rows of five and the number of seeds that germinated in each of the 120 rows are summarised below.

Number of germinating seeds	0	1	2	3	4	5
Number of rows	1	1	3	16	46	53

(a) Use the results in the table to determine

(i) the probability that no more than 4 seeds germinated in a randomly selected row. (1 mark)

Solution
$P(X \leq 4) = \dfrac{67}{120} \quad (= 0.558\overline{3})$
Specific behaviours
✓ correct probability

(ii) the mean number of seeds that germinated per row. (1 mark)

Solution
$\bar{x} = 4.2$
Specific behaviours
✓ correct mean

(b) Another row of five seeds is planted. Determine the probability that no more than 4 seeds germinate in this row if the number that germinate per row is binomially distributed with the above mean. (2 marks)

Solution
$5p = 4.2 \Rightarrow p = \dfrac{4.2}{5} = 0.84$ $Y \sim B(5, 0.84)$ $P(X \leq 4) = 0.5818$
Specific behaviours
✓ calculates p ✓ correct probability

Suppose it is known that 66% of all seeds planted will germinate and that seeds are now planted in rows of 16.

(c) Assuming that seeds germinate independently of each other, determine

(i) the most likely number of seeds to germinate in a row. (1 mark)

Solution
11 seeds
Specific behaviours
✓ correct number

(ii) the probability that at least 9 seeds germinate in a randomly chosen row. (2 marks)

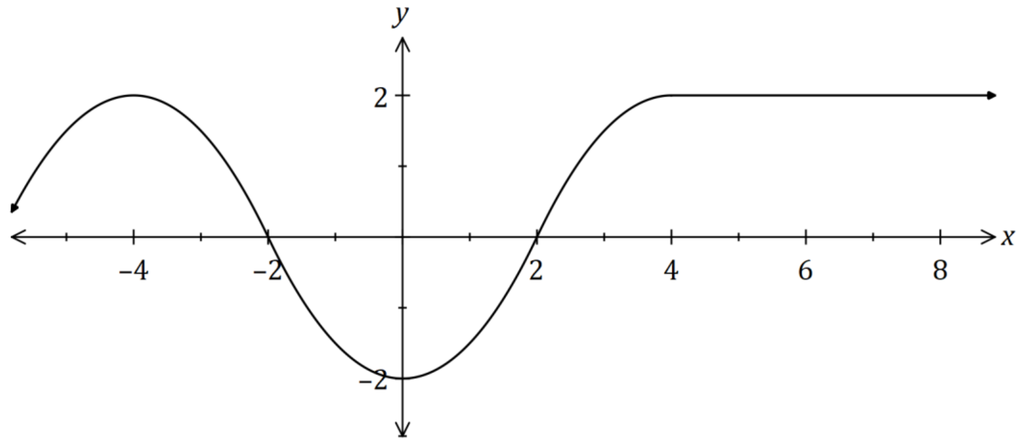
Solution
$W\sim B(16, 0.66)$ $P(W \geq 9) = 0.8609$
Specific behaviours
✓ states distribution ✓ correct probability

(iii) the probability that in eight randomly chosen rows, exactly six rows have at least 9 seeds germinating in them. (2 marks)

Solution
$V\sim B(8, 0.8609)$ $P(V = 6) = 0.2206$
Specific behaviours
✓ states distribution ✓ correct probability

Question 18 **(9 marks)**

The graph of $y = f(x)$ is shown below.



Let $A(x)$ be defined by the integral $A(x) = \int_{-4}^{x} f(t)\,dt$ for $x \geq -4$.

(a) Use the graph of $y = f(x)$ to identify all the turning points of the graph of $y = A(x)$, stating the x-coordinate and nature of each point. (2 marks)

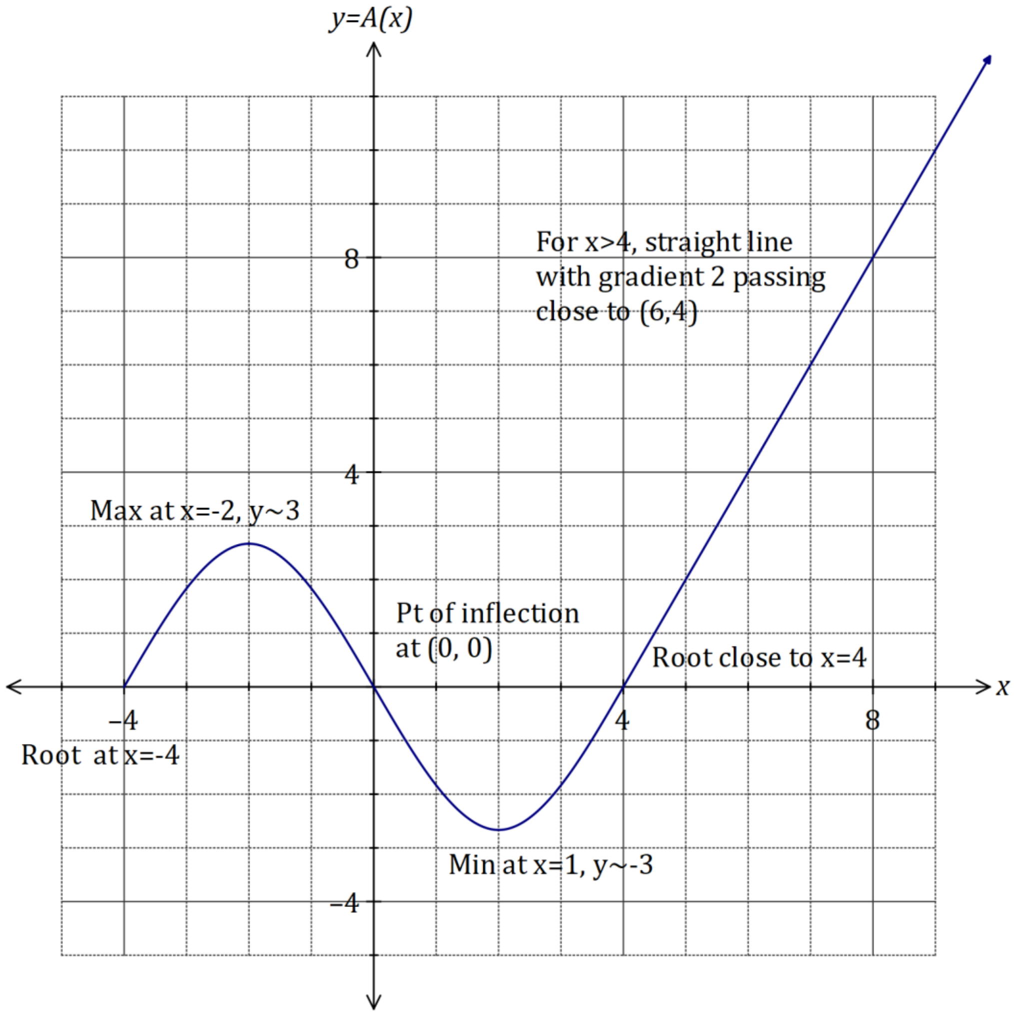
Solution
At $x = -2$ there is a maximum At $x = 2$ there is a minimum
Specific behaviours
✓ location of maximum ✓ location of minimum

It is also known that $A(4) = 0$.

(b) Using the graph of $y = f(x)$ or otherwise, explain why $A(7) = 6$. (2 marks)

Solution
$A(7) = A(4) + \int_4^7 f(x)dx$. From the graph $\int_4^7 f(x)dx = 2 \times 3 = 6$, and hence $A(7) = 0 + 6 = 6$.
Specific behaviours
✓ shows use of $A(4)$ and integral ✓ explanation, using area and $A(4)$

(c) Sketch the graph of $y = A(x)$ on the axes below, indicating and labelling the location of all key features. (5 marks)



Solution
See graph
Specific behaviours
✓ Labelled point of inflection at origin ✓ Labelled roots, as indicated ✓ Curve $-4 < x < 0$ with labelled maximum ✓ Curve $0 < x < 4$ with labelled minimum ✓ Straight line, as indicated

Question 19 **(7 marks)**

An aquarium, with a volume of $50\,000$ cm^3, takes the shape of a rectangular prism with square ends of side x cm and no top. The glass for the four vertical sides costs 0.05 cents per square cm and for the base costs 0.08 cents per square cm. The cost of glue to join the edges of two adjacent pieces of glass is 0.6 cents per cm. Assume the glass has negligible thickness and ignore any other costs.

(a) Show that $C = \frac{x^2}{1000} + \frac{9x}{250} + \frac{90}{x} + \frac{600}{x^2}$, where C is the cost, in dollars, to make the aquarium. (4 marks)

Solution
Let y be third length, so that $x^2y = 50000 \Rightarrow y = \frac{50000}{x^2}$ Cost of glass: $C_G = 0.05\left[2x^2 + 2x\left(\frac{50000}{x^2}\right)\right] + 0.08x\left(\frac{50000}{x^2}\right)$ Cost of edges: $C_E = 0.6\left[6x + 2\left(\frac{50000}{x^2}\right)\right]$ Total cost: $C = \frac{1}{100}(C_G + C_e) = \frac{x^2}{1000} + \frac{9x}{250} + \frac{90}{x} + \frac{600}{x^2}$
Specific behaviours
✓ expression for third side in terms of x ✓ cost of glass (simplification not required) ✓ cost of edges (simplification not required) ✓ sums and converts to dollars

(b) Show use of a calculus method to determine the minimum cost of making the aquarium. (3 marks)

Solution
$$\frac{dC}{dx} = \frac{x^4 + 18x^3 - 45000x - 600000}{500x^3}$$ $$\frac{dC}{dx} = 0 \text{ when } x = 34.48 \text{ cm}$$ $$C(34.48) = \$5.55$$
Specific behaviours
✓ shows marginal cost ✓ determines value of x so that marginal cost is zero ✓ determines minimum cost, to nearest cent.

Question 20 **(7 marks)**

A small body has displacement $x = 0$ when $t = 8$ and moves along the x-axis so that its velocity after t seconds is given by

$$v(t) = 10\sin\left(\frac{\pi t}{24}\right) \text{ cm/s}$$

(a) Determine an equation for $x(t)$, the displacement of the body after t seconds. (3 marks)

Solution
$x = -\dfrac{10 \times 24}{\pi}\cos\left(\dfrac{\pi t}{24}\right) + c$ $t = 0 \Rightarrow 0 = \dfrac{-240}{\pi}\cos\left(\dfrac{\pi}{3}\right) + c \Rightarrow c = \dfrac{120}{\pi}$ $x = \dfrac{120}{\pi} - \dfrac{240}{\pi}\cos\left(\dfrac{\pi t}{24}\right)$
Specific behaviours
✓ integrates v correctly ✓ attempts to find constant using substitution ✓ correct equation

(b) Describe, with justification, how the speed of the body is changing when $t = 32$. (4 marks)

Solution
$v(32) = 10\sin\left(\dfrac{4\pi}{3}\right) = -5\sqrt{3}$ $a = \dfrac{5\pi}{12}\cos\left(\dfrac{\pi t}{24}\right)$ $a(32) = \dfrac{5\pi}{12}\cos\left(\dfrac{4\pi}{3}\right) = -\dfrac{5\pi}{24}$ Since the body has a negative velocity and a negative acceleration then its speed is increasing when $t = 32$.
Specific behaviours
✓ clearly shows v is negative ✓ expression for a ✓ clearly shows a is negative ✓ explains increasing speed using signs of v and a

Question 21 **(7 marks)**

(a) Given that $f(t) = \sin\left(4t + \frac{\pi}{4}\right)$ and $F(x) = \int_0^x f(t)\,dt$, determine the exact value of

(i) $F\left(\frac{\pi}{8}\right)$. (1 mark)

Solution
$F\left(\dfrac{\pi}{8}\right) = \dfrac{\sqrt{2}}{4}$
Specific behaviours
✓ correct value

(ii) $F'\left(\frac{\pi}{8}\right)$. (2 marks)

Solution
$F'(x) = f(x)$ $f\left(\dfrac{\pi}{8}\right) = \dfrac{\sqrt{2}}{2}$
Specific behaviours
✓ recognises $F'(x) = f(x)$ ✓ correct value

(b) Given that $G(x) = \int_4^x g(t)\,dt$, $\frac{d^2G}{dx^2} = 6\sqrt{x} - 5$ and $G(9) = 174$, determine $g(t)$. (4 marks)

Solution
$G'(x) = g(x)$ $G''(x) = g'(x) = 4 + 3\sqrt{x}$ $g(x) = 4x^{1.5} - 5x + c$ $G(9) = \displaystyle\int_4^9 4t^{1.5} - 5t + c\,dt$ $= \dfrac{1751}{10} + 5c$ $\dfrac{1751}{10} + 5c = 174 \Rightarrow c = -\dfrac{11}{50}$ $g(t) = 4t^{1.5} - 5t - \dfrac{11}{50}$
Specific behaviours
✓ shows $G'(x) = g(x)$ ✓ uses $G''(x) = g'(x)$ to obtain $g(x)$ with constant c ✓ integrates again to obtain $G(9)$ ✓ evaluates constant c and writes expression for $g(t)$

End of questions

Supplementary page

Question number: _________